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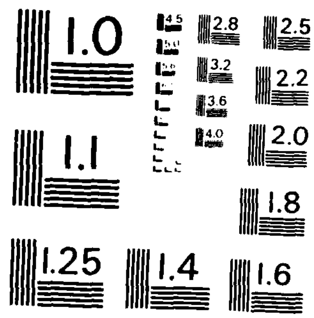
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APPLICATIONS OF SEMI-REGENERATIVE THEORY TO COMPUTATIONS

OF STATIONARY DISTRIBUTIONS OF MARKOV CHAINS

by

W. K. Grassmann and Michael I. Taksar

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Fourth Floor, Encina Hall
Stanford University
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94305

APPLICATIONS OF SEMI-REGENERATIVE THEORY TO COMPUTATIONS OF STATIONARY DISTRIBUTIONS OF MARKOV CHAINS*

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1. Introduction

Arguments from Regenerative Theory have been used by a number of authors to solve equilibrium equations in queueing problems. In particular, these arguments are prominent in the matrix-geometric solution pioneered by Neuts [1981], but they are also used by Kleinrock [1975] in order to analyze the GI/G/m queue and by Grassmann and Chaudhry [1982]. Since there is a number of applications for these methods, it seems appropriate to investigate them in further detail.

In this paper we use Semi-Regenerative Theory, which is a generalization and sophistication of Regenerative Theory. We believe that this is the first paper which uses Semi-Regenerative Theory for developing numerical (nonsimulation) algorithms to find the steady-state distribution of a Markov chain. The algorithm obtained is a modification of the Gauss-Jordan method, in which all the elements used in computations are always nonnegative, which makes the algorithm numerically stable.

To apply the theory in question to a given Markov chain ($Y_n, n = 0,1,\ldots$) we must represent the latter as a semi-regenerative process. To this end we consider a subset D of the state space of Y_n and we

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record successive visits of the chain Y_n to this subset. Let T_n, $n = 1,2,\ldots$, be the time of n-th visit to D and X_n be the position of the chain at T_n. Then the process (X_n,T_n) is a Markov Renewal process (see Çinlar [1975] Chapter 10) and Y_n is a semi-regenerative process with T_n being the semi-regenerative epochs and (X_n,T_n) being the imbedded Markov Renewal process.

The latter means that if we consider the "cycles"

$$A_n = \{Y_{T_{n-1}}, Y_{T_{n-1}+1}, \ldots, Y_{T_n - 1}\}$$

then the conditional distribution of the sequence $(A_n, A_{n+1}, \ldots)$ given the past of the process up to T_{n-1}, depends only on X_{n-1}; and all A_n are conditionally independent given $(X\cdot, T\cdot)$.

The analysis of the behavior of $Y_\cdot$ from one semi-regenerative epoch to another produces the main relation between the steady-state probabilites that is used for developing the algorithm.

2. Proof of the Main Result

We consider an irreducible aperiodic positive recurrent Markov chain Y_n with a state space $E = \{0,1,2,\ldots\}$. It is known (see Çinlar [1975], Chapter 8) that such a chain reaches a steady state, i.e.,

$$P_i\{Y_n = j\} \to p_j \quad , \quad j = 0,1,\ldots \quad ,$$

or in a shorter version

$$Y_n \Rightarrow Y$$

where Y is an integer-valued random variable with distribution $(p_0, p_1, \dots)$. Here $P_i\{\cdot\} \equiv P\{\cdot \mid Y_0 = i\}$. The notation E_i must be understood in a similar way. Let

$$D = \{0,1,2,\dots, d-1\}$$

and

(2.1) $$T = \min\{m > 0:\ Y_m \in D\}$$

Put

(2.2) $$v_{ki}^{(d)} = E_k\{\sum_{m=0}^{T-1} 1_i(Y_m)\} = E_k\{\#m:\ m < T \text{ and } Y_m = i\}$$

Theorem: Let $(p_0, \dots, p_n)$ be the steady-state distribution of the Markov chain Y_n. Then

(2.3) $$p_i = \sum_{k=0}^{d-1} v_{ki}^{(d)} p_k$$

Proof: Consider the process $Z_t = Y_{[t]}$, where $[t]$ is the integer part of t. Let

$$T_1 = T$$

$$T_{n+1} = \min\{m > T_n:\ Y_m \in D\}$$

(2.4) $$X_n = Y_{T_n} \equiv Z_{T_n}$$

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Then for each $n \geq 1$ we have

(a) T_n is a stopping time for Z.

(b) X_n is determined by $\{Z_u;\ u \leq T_n\}$

(c) for each function f defined on E^m and each

$t_1 < t_2 < \ldots < t_m$

$$E_i\{f(Z_{T_n+t_1},\ldots,Z_{T_n+t_m})|Z_u;\ u \leq T_n\}$$

$$= E_j\{f(Z_{t_1},\ldots,Z_{t_m})\} \quad \text{on the set} \quad \{X_n = j\}$$

Really, (a) follows from the definition of T_n, (b) is a consequence of (2.4) and (c) is due to the strong Markov property of Y_n.

In particular X_n is a Markov chain in the state space $D = \{0,1,\ldots, d-1\}$. This chain is irreducible because the original chain Y_n is irreducible. Let ν_i, $i = 1,2,\ldots, d-1$ be the (unique) invariant distribution for Y_n. Let

$$m(i) = E_i\{T\}$$

and

$$K_t(j,i) = P_j\{Z_t = i,\ T > t\} \equiv P_j\{Y_{[t]} = i;\ Y_k \quad D, \text{ for } k < T\}$$

According to Çinlar [1975], Chapter 10, Theorem (6.12)

$$\text{(2.5)} \qquad \lim_{t\to\infty} P_i\{Z_t = j\} = \Big(\sum_{\ell\in D} m(\ell)\nu_\ell\Big)^{-1} \sum_{k\in D} \nu_k \int_0^\infty K_t(k,j)\,dt \quad .$$

We must mention that (2.5) was proved in Çinlar [1975] under the assumption that (X,T) is a positive recurrent, irreducible and aperiodic Markov renewal process. In our case, (X,T) is irreducible and positive recurrent due to similar properties of Y_n, but (X,T) is not aperiodic. The random variables T_n are always integer valued. Since Y_n is aperiodic (as a discrete Markov chain) then (X,T) is periodic with period one. However, the proof of (2.5) in Çinlar [1975] goes through without change for the process Z such that $Z_t = Z_{[t]}$ and such that the period of the imbedded Markov Renewal process is 1.

The function $K_t(k,j)$ is piecewise constant, namely $K_t(\cdot,\cdot) = K_{[t]}(\cdot,\cdot)$ hence (using Fubini's theorem below)

$$\int_0^\infty K_t(k,j)dt = \sum_{m=0}^\infty K_m(k,j) \tag{2.6}$$

$$= \sum_{m=0}^\infty P_k\{Z_m = j,\ T > m\} = \sum_{m=0}^\infty P_k\{Y_m = j,\ T > m\}$$

$$= \sum_{m=0}^\infty E_k\{1_j(Y_m)1_{T>m}\} = E_k\{\sum_{m=0}^{T-1} 1_j(Y_m)\} = v_{kj}^{(d)}$$

On the other hand

$$\lim_{t\to\infty} P_i\{Z_t = j\} = \lim_{t\to\infty} P_i\{Z_{[t]} = j\} = \lim_{m\to\infty} P_i\{Y_m = j\} = p_j \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we have

$$p_j = \Big(\sum_{\ell\in D} \nu_\ell m(\ell)\Big)^{-1} \sum_{k\in D} \nu_k v_{kj}^{(d)} \tag{2.8}$$

Let us apply (2.8) for $j \in D$. It is easy to see that if $k,j \in D$ then ,

$$v_{kj}^{(d)} = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}$$

Therefore, for $j \in D$ formula (2.8) becomes

$$p_j = \frac{\nu_j}{\sum_{\ell \in D} \nu_\ell m(\ell)} \tag{2.9}$$

Now, substituting (2.9) into (2.8), we get (2.3).

The next proposition describes the relations between $v_{kj}^{(d)}$ for different k,j and d. Its significance will become clear later when we develop a numerical algorithm.

<u>Proposition</u>: Let $v_{kj}^{(d)}$ be given by (2.2), and let p_{ij} be the transition probabilities of the Markov chain Y_n. Then

$$v_{kj}^{(d)} = v_{kj}^{(d+1)} + v_{dj}^{(d+1)} v_{kd}^{(d)} \quad , \quad k \neq d \tag{2.10}$$

$$v_{kd}^{(d)} = p_{kd} + \sum_{j>d} v_{jd}^{(d)} p_{kj} \quad , \quad k \neq d \tag{2.11}$$

$$v_{kj}^{(d)} = v_{kj}^{(j)} + \sum_{m=d}^{j-1} v_{km}^{(d)} v_{mj}^{(j)} \quad , \quad j > d \tag{2.12}$$

<u>Proof</u>:

1) Let D consist of the first d points from 0 to $d-1$ and let T be defined by (2.1). Let τ be the first hitting time of the point d. Then we can write

$$v_{kj}^{(d)} = E_k\{\sum_{n=0}^{T-1} 1_j(Y_n)\}$$
$$= E_k\{\sum_{n=0}^{T\wedge\tau-1} 1_j(Y_n)\} + E_k\{\sum_{n=T\wedge\tau}^{T-1} 1_j(Y_n)\} \tag{2.13}$$

The first term in the right-hand side of (2.13) is obviously equal to $v_{kj}^{(d+1)}$. Next, we must notice that if $\tau > T$ then the second expression in the right-hand side of (2.13) equals zero. On the set $\tau < T$ we can use the strong Markov property to obtain

$$E_k\{\sum_{n=T\wedge\tau}^{T-1} 1_j\{Y_n\}\} = E_k\{v_{dj}^{(d)} \quad \tau < T\} = v_{dj}^{(d)}P_k\{\tau < T\} \tag{2.14}$$

Substituting (2.14) into (2.13), we get

$$v_{kj}^{(d)} = v_{kj}^{(d+1)} + v_{dj}^{(d)}P_k\{\tau < T\} \tag{2.15}$$

Now, iterate (2.15), namely, put $k = d$ in (2.15) and substitute the value of $v_{dj}^{(d)}$ in the right-hand side of (2.15). We get

$$v_{kj}^{(d)} = v_{kj}^{(d+1)} + P_k\{\tau < T\}[v_{dj}^{(d+1)} + v_{dj}^{(d)}P_d\{\tau < T\}] \tag{2.16}$$

Iterating (2.15) n times, we get

$$v_{kj}^{(d)} = v_{kj}^{(d+1)} + P_k\{\tau < T\}[\sum_{m=0}^{n-1} v_{dj}^{(d+1)}(P_d\{\tau < T\})^m$$
$$+ (P_d\{\tau < T\})^n v_{dj}^{(d)}] \tag{2.17}$$

If $P_d\{\tau < T\} = 1$ then, using the strong Markov property, we get that $P_d\{T < \infty\} = 0$, which contradicts the assumption that Y_n is irreducible. Therefore, $P_d\{\tau < T\} < 1$ and we can pass to the limit in (2.17) as $n \to \infty$.

$$v_{kj}^{(d)} = v_{kj}^{(d+1)} + v_{dj}^{(d+1)} \sum_{m=0}^{\infty} P_k\{\tau < T\}(P_d\{\tau < T\})^m \tag{2.18}$$

Using the strong Markov property once more, we see that if $k \neq d$, then

$$\sum_{m=0}^{\infty} P_k\{\tau < T\}(P_d\{\tau < T\})^m = \sum_{m=0}^{\infty} P_k\{\sum_{\ell=0}^{T-1} 1_d(Y_\ell) > m\} \tag{2.19}$$

$$= E_k\{\sum_{\ell=0}^{T-1} 1_d(Y_\ell)\} = v_{kd}^{(d)}$$

Combining (2.18) and (2.19), we get (2.10).

2) To prove (2.11), write (below we use the fact that $T = 1$ on the set $Y_1 \in D$)

$$v_{kd}^{(d)} = E_k\{\sum_{m=1}^{\infty} 1_d(Y_m)1_{T>m}\} = E_k\{1_d(Y_1)1_{T>1}\} \tag{2.20}$$

$$+ E_k\{1_{E\setminus D}(Y_1)E_k\{\sum_{m=2}^{\infty} 1_d(Y_m)1_{T>m}|Y_1\}\}$$

If $Y_1 = d$ then obviously $T > 1$; therefore, the first term in the right-hand side of (2.20) equals

$$E_k\{1_d(Y_1)1_{T>1}\} = E_k\{1_d(Y_1)\} = p_{kd}$$

Due to the Markov property, the conditional expectation in the second term in the right-hand side of (2.20) equals

$$E_{Y_1}\{\sum_{m=1}^{\infty} 1_d(Y_m)1_{T>m}\} = v_{Y_1 d}^{(d)}$$

and hence (2.20) becomes

$$v_{kd}^{(d)} = p_{kd} + E_k\{v_{Y_1 d}^{(d)} 1_{E\setminus D}(Y_1)\}$$

$$= p_{kd} + \sum_{j>d} v_{jd}^{(d)}$$

3) Formula (2.12) generaliz(. .0), its proof and the ideas behind the proof are similar to those of (2.10), but require more computations, and we omit them here.

3. An Algorthm to Find Equilibrium Probabilities

In this section, we present an algorithm to find the equilibrium probabilities p_i, which will then be interpreted in terms of the $v_{ij}^{(j)}$. This interpretation is interesting for its own sake, but it can also be exploited in a number of ways as will be shown.

The equilibrium probabilities p_i are given by the steady state equations, that is

$$0 = \sum_{i=0}^{N} p_i p_{ij} - p_j = \sum_{i=0}^{N} p_i(p_{ij} - \delta_{ij}) \quad , \quad j = 0,1,\ldots,N \quad .$$

Here δ_{ii} is 1 if $i = j$ and zero otherwise. To find the p_i, we proceed as follows. We solve the Nth equation for p_N, and eliminate p_N from all other equations. Then, we solve equation $N - 1$ for p_{N-1}, and eliminate p_{N-1} from all other equations, except from equation

N. We continue this way until we solve the first equation for p_1. In other words, we always use the diagonal element as pivot, and we apply Gaussian elemination, starting with equation N and ending with equation 1. If a^n_{ij} are the values obtained before solving for p_n, one has

$$a^N_{ij} = p_{ij} - \delta_{ij}$$

and

$$(3.1) \qquad a^{n-1}_{in} = -\frac{a^n_{in}}{a^n_{nn}} , \quad 0 \leq i < n$$

$$(3.2) \qquad a^{n-1}_{ij} = a^n_{ij} + a^n_{nj} a^{n-1}_{in} , \quad 0 \leq i < n , \quad 0 \leq j < n$$

$$(3.3) \qquad a^{n-1}_{ij} = a^n_{ij}$$

This method can only be applied if all $a^n_{nn} > 0$. This is where the theory presented earlier is helpful. Clearly, the elimination procedure gives

$$p_j = \sum_{i=0}^{j-1} p_i a^n_{ij} , \quad j > n .$$

It is also clear that the a^n_{ij} are uniquely determined by the p_{ik}, $k \geq j$. On the other hand, we have equation (2.3), which implies

$$p_j = \sum_{i=0}^{j-1} p_i v^{(j)}_{ij} .$$

Here, the $v_{ij}^{(n)}$ are similarly determined uniquely by p_{ik}, $k > j$. Hence

$$a_{ij}^{n} = v_{ij}^{(j)} \quad , \quad j > n \quad . \tag{3.4}$$

This implies that $0 \leq a_{ij}^{n} < \infty$, $j > n$. This, together with (3.1) implies that a_{nn}^{n} cannot be equal zero. We can even say more. Indeed, one has

$$a_{nn}^{n} = - \sum_{j=0}^{n-1} a_{nj}^{n} \quad . \tag{3.5}$$

To prove (3.5), we show by complete induction that

$$\sum_{j=0}^{n} a_{ij}^{n} = 0 \tag{3.6}$$

Clearly, this equation holds for $n = N$. We now prove (3.6) for $n - 1$. Because of (3.3), we only need to consider the case where $j < n$. Equations (3.1) and (3.2) then give

$$\sum_{j=0}^{n-1} a_{ij}^{n-1} = \sum_{j=0}^{n-1} \left(a_{ij}^{n} - \frac{a_{nj}^{n} a_{in}^{n}}{a_{nn}^{n}} \right)$$

$$= \sum_{j=0}^{n-1} a_{ij}^{n} - \sum_{j=0}^{n-1} a_{ij}^{n} \frac{a_{in}^{n}}{a_{nn}^{n}}$$

$$= \sum_{j=0}^{n-1} a_{ij}^{n} + a_{in}^{n} = \sum_{j=0}^{n} a_{ij}^{n} = 0 \quad .$$

This proves (3.6), and with it (3.5).

At this point, it should become apparent that the elimination suggest.. by (3.1) to (3.3) can be done without ever using subtractions or negative numbers. We merely need to replace a_{nn}^{n} in (3.1) by (3.5), giving

$$a_{in}^{n-1} = \frac{a_{in}^{n}}{\sum_{j=0}^{n-1} a_{nj}^{n}} , \quad 0 \leq i < n \tag{3.7}$$

and omit (3.2) for the case that $i = j$. Equations (3.2) and (3.7) involve then only a_{ij} with $i \neq j$, and these will always remain positive, given they were positive to start with.

Because of (3.4), the elimination described by (3.2) and (3.7) gives for $n = 0$

$$p_j = \sum_{i=0}^{j-1} p_i a_{ij}^{0} = \sum_{i=0}^{j-1} p_i v_{ij}^{(j)} , \quad j = 1,2,\dots,N . \tag{3.8}$$

From (2.12), we find

$$v_{0j}^{(1)} = v_{0j}^{(j)} + \sum_{k=1}^{j-1} v_{0k}^{(1)} v_{kj}^{(j)} . \tag{3.9}$$

Relation (3.9) allows one to find $v_{0j}^{(1)}$ recursively, once $v_{0j}^{(j)}$ is given. Once all $v_{0j}^{(1)}$ are found, p_0 becomes

$$p_0 = \frac{1}{\left(1 + \sum_{j=1}^{N} v_{0j}^{(1)}\right)}$$

(3.10)

$$p_j = p_0 v_{0j}^{(1)} , \quad j > 0 .$$

With this, all p_j are found, and we can now count how many operations are needed to find all p_j, $j = 0,1,\ldots,N$.

We note that for each n, $\sum a_{nj}$ has to be calculated, which gives n operations, equation (3.7) must be applied n times, requiring another n operations, and equation (3.2) is applied n^2 times, giving another $2n^2$ operations. Since n varies from 1 to N, this gives (see also E. Isaacson, H. B. Keller [1966])

$$\sum_{n=1}^{N} 2(n^2 + n) = 2\sum_{n=1}^{N} n(n + 1) = \frac{2N(N + 1)(N + 2)}{3}$$

operations. The number of operations is thus of order $2N^3/3$.

Frequently, there exist g,h such that $p_{ij} = 0$ for $j < i - g$ and $j > i + h$. The a^n_{ij} inherit this property as can be seen by complete induction. To find the demoninator of (3.7) then requires only $\min(n,g)$ additions. Once the denominator is found, (3.7) needs only be applied $\min(n,h)$ times, which gives a total of $\min(n,h)$ multiplications. (3.2) similarly requires a total of $2 \min(n,h) \cdot \min(n,g)$ operations. Since n runs from 1 to N, this gives order $2Nhg$ operations, which is considerably less than $2N^3/3$ equations as long as h or g is small compared to N. The number of operations to do (3.9) and (3.10) is always relatively insignificant.

Sometimes, the $v^{(d)}_{kj}$, $j \neq 1$ and $j \neq d$ are needed as well. They are easily found recursively from the $v^{(j)}_{kj}$, using equation (2.12). Moreover, by substituting $v^{(d)}_{ki}$ in (2.3) from (2.11), we find the following interesting equation

$$p_j = \sum_{k=0}^{d-1} p_{kj}(p_{kj} + \sum_{j>d} p_{ij} v_{kj}^{(d)}) \quad , \quad j = 0,1,\ldots, d-1 \quad . \tag{3.11}$$

Thus, once the $v_{kj}^{(d)}$ are found, we have j equations, and $(j - 1)$ independent equations, which allow one to find p_j, $j = 0,1,\ldots, d - 1$, except for a factor. Applications of (3.11) will be given in the next section.

4. Applications

To illustrate the method, we now consider a number of special cases and discuss some numerical results. Some of the examples discussed deal with continuous-time Markov processes. This is no major problem because all the methods carry over to continuous-time Markov processes, provided p_{ij}, $i \neq j$ is replaced by the rate a_{ij}. Also, the $v_{ij}^{(d)}$ can be interpreted similarly. The diagonal elements, that is, the $p_{ii} - 1$, respectively, the a_{ii}, are irrelevant. We can set these elements equal to 0.

In most cases considered here, p_{ij} (respectively a_{ij}) are zero for $j > i + h$ and $j < i - g$. In this case, one finds from (3.2), (3.4) and (3.7)

$$v_{in}^{(n)} = \begin{cases} 0 & i < n - h \\ \dfrac{a_{in}^{n}}{\sum\limits_{j>0,n-g}^{n-1} a_{n,j}^{n}} & 0, n - h \leq i < n \end{cases} \tag{4.1}$$

$$
(4.2) \qquad a_{ij}^{n-1} = \begin{cases} a_{ij}^{n} + a_{nj}^{n} v_{in}^{(n)} & 0, n-h \le i < n \\ & 0, n-g \le j < n, j \ne i \\ a_{ij}^{n} & \text{otherwise} \end{cases}
$$

It can now immediately be seen that in the case of a birth-death process, where $a_{i,i+1} = \lambda_i$ and $a_{i,i-1} = \mu_i$, $a_{ij} = 0$, $j > i + 1$ or $j < i - 1$, (4.2) will not be used at all, and (4.1) becomes

$$
v_{n-1,n}^{(n)} = \frac{\lambda_{n-1}}{\mu_n}
$$

Relation (3.8) then leads to the well-known relationship

$$
p_j = p_{j-1} \frac{\lambda_{j-1}}{\mu_j} .
$$

This result gives a nice check for our derivations. We also note that our method does not pick the (less efficient) simple recursive solution of the steady state equations.

Next, consider the $E_2/M/1$ queue, with the additional restriction that the number of "phases" in the system is restricted to N. In this case, $a_{ij} = \mu$ for $j = i - 2$, $i = 2,3,\ldots,N$ and $a_{ij} = 2\lambda$ for $j = i + 1$, $i = 0,1,\ldots, N - 1$. All other a_{ij}, $i \ne j$ are zero.

Since $h = 1$, (4.1) is only applied for $i = n - 1$, that is

$$
(4.3) \qquad v_{n-1,n}^{(n)} = \frac{2\lambda}{a_{n,n-1}^{n} + \mu}
$$

Equation (4.2) is similarly applied only for $i = n - 1$, $j = n - 2$, which gives

(4.4) $$a^{n-1}_{n-1,n-2} = a^{n}_{n-1,n-2} + a^{n}_{n,n-2}v^{(n)}_{n-1,n} = \mu v^{(n)}_{n-1,n}$$

Equations (4.3) and (4.4) can be combined to give

$$v^{(n)}_{n-1,n} = \frac{2\lambda}{\mu v^{(n+1)}_{n,n+1} + \mu} = \frac{\frac{2\lambda}{\mu}}{1 + v^{(n+1)}_{n,n+1}} .$$

Moreover, it is easily verified that $v^{(N)}_{N-1,N} = 2\lambda/\mu$. This means that $v^{(n)}_{n-1,n}$ is really the beginning of a continued fraction

$$v^{(N-1)}_{N-2,N-1} = \frac{\frac{2\lambda}{\mu}}{1 + \frac{2\lambda}{\mu}}$$

$$v^{(N-2)}_{N-3,N-2} = \frac{\frac{2\lambda}{\mu}}{1 + \frac{\frac{2\lambda}{\mu}}{1 + \frac{2\lambda}{\mu}}}$$

and so on. In Table 1, the $v^{(n)}_{n-1,n}$ are listed for $N = 10$, $\mu = 1$ and $\lambda = 0.5$ and 1. It is clearly seen that the $v^{(n)}_{n-1,n}$ converge rather quickly. Then, due to (3.8),

$$p_n = v^{(n)}_{n-1,n}p_{n-1} .$$

Of course, our algorithm can solve much more complex problems than the ones presented above. To show this, we calculated the stationary probabilities for a number of cases. These results are given in Table 2.

Table 1: The $v_{n-1,n}^{(n)}$ for the $E_2/M/1$ queue

n	$\lambda = 0.5$ $v_{n-1,n}^{(n)}$	$\lambda = 1$ $v_{n-1,n}^{(n)}$
9	1	2
8	0.5	0.6667
7	0.6667	1.2
6	0.6	0.9090
5	0.625	1.0476
4	0.6154	0.9767
3	0.6190	1.0118
2	0.6176	0.9941
1	0.6182	1.0029

Table 2: Computational results

N	h	g	Execution Time*	Average Difference	Largest Difference	Largest Rel. Difference
100	10	10	1.6	2 E - 10	5 E - 10	7 E - 8
1000	10	10	15	5 E - 11	2 E - 10	2 E - 7
100	20	10	2.7	4 E - 10	5 E - 9	7 E - 8
100	10	20	2.7	2 E - 10	2 E - 9	7 E - 8

*DEC 2050 Basic, SCORE-System, Stanford University

In each case, the a_{ij} where generated randomly, using a uniform distribution between zero and 1. To find out how accurate the results are, we solved each problem twice, except that in the second run, state 0 became state N, state 1 became state N - 1 and so on. Absolute and relative deviations between these two methods are also given in Table 2. The execution time is the time per run in cpu seconds.

Applications are not limited to numerical investigations as the following examples indicate.

Kleinrock [1975] investigated the imbedded Markov process of the GI/M/m queue as follows. He defined σ_{d-1} to be the expected number of visits of state d between two visits of state d - 1. In the process in question, it is impossible to visit state d from the states d - 2, d - 3,..., which implies that σ_d is also equal to the number of visits to state d during a stay among the states d, d + 1, d + 2,.... In symbols, h = 1, and consequently

$$\sigma_d = v_{d-1,d}^{(d)} .$$

Kleinrock proves now that

$$p_d = \sigma_{d-1} \cdot p_{d-1} ,$$

a fact that immediately follows from (4.1) with h = 1. Kleinrock argues further that σ_{d-1} is independent of d as soon as $p_{ij} = p_{j+k,j}$, $h > 0$ is independent of j. This also follows easily from our derivation, because $v_{d-1,d}^{(d)}$ is determined by the $p_{j+k,j}$, $j > d$. More complex systems than Kleinrock's can be analyzed in a similar way. In particular, our methods can give new insights into the matrix-geometric solution of Neuts [1981] as the reader may verify.

Next, we present two applications of the equation (3.11). The first application concerns a system with a huge number of states, where the probabililty mass is concentrated in a few states only, say in the states $0,1,2,\ldots, d-1$. Such systems are encountered frequently in connection with queueing networks at a low traffic intensity. In such systems, we can obtain the $v_{kj}^{(d)}$ by simulation, and caluclate the p_j, $j < d$ from (3.11), and the p_j, $j \geq d$ from (2.3).

The second application of (3.11) concerns a group of Markov processes, in which the p_{ij}, $j \geq d$ are identical, but the p_{ij}, $j < d$ are not. Since the $v_{ij}^{(d)}$ depend only on p_{ij}, $j \geq d$, this means that the $v_{ij}^{(d)}$ are identical for all these problems, and need not be recalculated. For this case, (3.11) provides a convenient way to solve such problems. This is of importance in the case of sensitivity analysis.

5. Conclusions

It has been shown (Grassmann [1983]) that algorithms, dealing with non-negative elements only, and containing no subtractions, are extremely resistant to rounding errors. In this paper, we derived such an algorithm. The algorithm combines elements from regenerative theory with algebraic methods. The algorithm is still effective for problems with a size of $N = 1000$, a range where normal Gaussian elimination tends to fail because of the accumulation of rounding errors. The algorithm also lends itself to exploit band-structures which are prevalent in many queueing problems.

Because of the interpretation of some key elements of the algorithm, using regenerative theory, the algorithm can also make theoretical contributions. In particular, it makes an argument of Kleinrock more transparent. We also mentioned that it can be used in connection with the matrix-geometric solutions of Neuts. Finally, the regenerative underpinnings of our algorithm allow one to combine analytical and simulation methods.

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